

Rusty Scatter Branes

Ramzi R. Khuri^{a,b,1} and Robert C. Myers^{b,2}

^a *Theory Division, CERN, CH-1211
Geneva 23, Switzerland*

^b *Department of Physics, McGill University
Montréal, Québec, Canada H3A 2T8*

Abstract

We derive double dimensional reduction/oxidation in a framework where it is applicable to describe general non-static (and anisotropic) p -brane solutions. Given this procedure, we are able to relate the dynamical interaction potential for parallel extremal p -branes in D dimensions to that for extremal black holes in $D - p$ dimensions. In particular, we find that to leading order the potential vanishes for all κ -symmetric p -branes.

CERN-TH/95-331
McGill/95-61
December 1995

¹khuri@nxth04.cern.ch

²rcm@hep.physics.mcgill.ca

1 Introduction

Recent investigations in nonperturbative string theory are beginning to reveal new connections between different string theories [1]. In fact many different string theories appear to be different phases of a single underlying theory [2]. Within these discussions, it appears that extended objects, other than just strings, play an important role in establishing the full web of connections relating various string theories. Hence these exciting new results have generated a renewed interest in p -branes (*i.e.*, p -dimensional extended objects) and their interactions.

Many different background field solutions have been constructed describing extended objects of different dimensions in various theories (see [3] and references therein). Typically these solutions involve a single d -form potential and a scalar field, *i.e.*, dilaton, coupled to gravity. Further, the solutions may be characterized as being static and isotropic [4, 5] — in fact, the latter is used to refer to the fact that the solutions are Lorentz invariant in the directions parallel to the world-volume of the p -brane. With a d -form potential in D dimensions, one naturally finds two dual isotropic p -branes. The first with $p = d - 1$ carries an “electric” charge from the $(d + 1)$ -form field strength. The dual object with $p = D - d - 3$ carries an analogous “magnetic” charge.

In certain cases, the p -branes satisfy an extremality or zero-force condition, and then two (or more) parallel p -branes can sit in static equilibrium. In the latter case, though, one expects that the precise cancellation of forces no longer holds when the p -branes are in motion, and so they will scatter in a nontrivial way. Only in very special cases would the scattering be trivial. In the past, there have been limited investigations of such scattering for particular p -branes [6, 7, 8]. The present paper provides a general analysis for the scattering of p -branes in any dimension for a broad class of theories.

Our approach begins by generalizing the usual reduction/oxidation procedure which relates p -branes in D dimensions to $(p + 1)$ -branes in $D + 1$ dimensions, as described in section 2. Our discussion is quite general and in particular not restricted to static or isotropic solutions. Thus this procedure can accomodate a discussion of moving (parallel) p -branes, as required to investigate scattering. As briefly reviewed in section 3, the scattering of extremal p -branes is determined by a computation of the metric on the moduli space. However, combining these calculations with our oxidation procedure, we can relate scattering of p -branes to that of extremal black holes. Thus the metric on the moduli space of extremal p -branes is simply related to that calculated by Shiraishi [9] for extremal black holes. We conclude with a discussion of our results in section 4.

2 Reduction and Oxidation

Double-dimensional reduction is a procedure relating one class of $(p+1)$ -branes in a $(D+1)$ -dimensional theory to another of p -branes in D dimensions. The inverse procedure going from D to $D + 1$ dimensions has come to be known as “oxidation”. Double-dimensional reduction was originally used to relate the $D = 11$ supermembrane action to that for

the Type IIa superstring in $D = 10$ [10]. The procedure was extended to generate new solutions in general dimensions in Refs. [5, 11], but their analysis was limited to certain supersymmetric (static and isotropic) solutions. Here our derivation of double dimensional reduction provides a straightforward extension of the previous discussions in that it is relevant for non-static p -brane solutions in theories with an arbitrary dilaton coupling constant. Such solutions will be relevant in the following discussion of p -brane scattering. We also point out that this procedure can be used to generate anisotropic p -brane solutions, which do not have the usual Lorentz invariance in the brane directions. As an example, the most general extremal anisotropic brane solution of the class of actions described below is presented in Appendix A.

The basic approach is to start with a $(p + 1)$ -brane solution in $D + 1$ dimensions, and eliminate one of the directions parallel to the brane to produce a p -brane in D dimensions. We begin with the action:

$$I = \frac{1}{16\pi\hat{G}} \int d^{D+1}x \sqrt{-\hat{g}} \left[\hat{R} - \frac{\hat{\gamma}}{2} (\nabla\hat{\phi})^2 - \frac{1}{2(d+2)!} e^{-\hat{a}\hat{\gamma}\hat{\phi}} \hat{F}^2 \right] \quad (1)$$

where $\hat{\gamma} = 2/(D-1)$.³ Here \hat{F} is a $(d+2)$ -form field strength defined in terms of a $(d+1)$ -form potential \hat{A} — *i.e.*, $\hat{F} = d\hat{A}$. In the following, it will be useful to define $\tilde{d} = D - d - 2$. Since we wish to allow the consideration of anisotropic branes, we will not restrict our discussion to the choices $p = d - 1$ or $\tilde{d} - 2$ for the $(p + 1)$ -branes in $D + 1$ dimensions.

We require that in the solutions all of the fields are independent of (at least) one of the spatial coordinates, denoted y , which runs parallel to the $(p + 1)$ -brane. This coordinate will be the direction which is removed in the double-dimensional reduction. Further we will require that in the $D + 1$ dimensional solution all of the nonvanishing components of the \hat{A} potential carry a y index. In the dimensionally reduced theory, we will have a d -form potential A with

$$A_{\mu\cdots\nu} = \hat{A}_{\mu\cdots\nu y}$$

and a corresponding $(d + 1)$ -form field strength $F = dA$. We also make a Kaluza-Klein reduction of the metric which is required to have the form

$$\hat{g}_{\mu\nu} = \begin{pmatrix} \bar{g}_{\mu\nu} & 0 \\ 0 & \exp[2\rho] \end{pmatrix}$$

where $\exp[2\rho]$ is the component \hat{g}_{yy} . One finds that $\hat{R}(\hat{g}) = \bar{R}(\bar{g}) - 2\bar{\nabla}^2\rho - 2(\bar{\nabla}\rho)^2$. Thus the gravity part of the action becomes

$$\int d^{D+1}x \sqrt{-\hat{g}} \hat{R} = L \int d^Dx \sqrt{-\bar{g}} e^\rho \bar{R}$$

where L is the volume of the compactified dimension. To remove the exponential factor in this action, we make a further conformal transformation: $\bar{g}_{\mu\nu} = \exp[-2\rho/(D-2)]g_{\mu\nu}$. The

³We choose a slightly unconventional normalization for the dilaton kinetic term because it provides some simplification of the formulae appearing in the discussion of p -brane scattering.

Ricci scalar then becomes

$$\bar{R} = \exp \left[\frac{2\rho}{D-2} \right] \left(R + \frac{D-1}{D-2} \nabla^2 \rho - \frac{D-1}{D-2} (\nabla \rho)^2 \right)$$

and now the full dimensionally reduced action is

$$I = \frac{L}{16\pi\hat{G}} \int d^D x \sqrt{-g} \left[R - \frac{D-1}{D-2} (\nabla \rho)^2 - \frac{\hat{\gamma}}{2} (\nabla \hat{\phi})^2 - \frac{1}{2(d+1)!} e^{-\hat{a}\hat{\gamma}\hat{\phi}} e^{-2\tilde{d}\rho/(D-2)} F^2 \right].$$

From this action, the scalar field equations of motion are

$$\begin{aligned} 2\frac{D-1}{D-2} \nabla^2 \rho + \frac{2\tilde{d}}{D-2} \left(\frac{1}{2(d+1)!} e^{-\hat{a}\hat{\gamma}\hat{\phi}} e^{-2\tilde{d}\rho/(D-2)} F^2 \right) &= 0 \\ \nabla^2 \hat{\phi} + \hat{a} \left(\frac{1}{2(d+1)!} e^{-\hat{a}\hat{\gamma}\hat{\phi}} e^{-2\tilde{d}\rho/(D-2)} F^2 \right) &= 0. \end{aligned}$$

Combining these equations, we find that $\nabla^2(\hat{\phi} - (D-1)\hat{a}/\tilde{d} \rho) = 0$, and so this linear combination represents a free scalar field which we have the liberty to set to zero. Hence setting $\rho = [\tilde{d}/((D-1)\hat{a})]\hat{\phi}$ in the above action, we find the kinetic term of the remaining scalar has an unconventional normalization. Thus we scale this field to define the canonical dilaton ϕ of the dimensionally reduced theory

$$\frac{\gamma}{2} (\nabla \phi)^2 = \left(\frac{\hat{\gamma}}{2} + \frac{\tilde{d}^2}{(D-2)(D-1)\hat{a}^2} \right) (\nabla \hat{\phi})^2$$

where $\gamma = 2/(D-2)$. With these choices the final action becomes

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left[R - \frac{\gamma}{2} (\nabla \phi)^2 - \frac{1}{2(d+1)!} e^{-a\gamma\phi} F^2 \right] \quad (2)$$

where $a^2 = [(D-2)/(D-1)]\hat{a}^2 + \tilde{d}^2/(D-1)$ and $G = \hat{G}/L$.

We can reverse this reduction process to construct an oxidation prescription as follows: In the D -dimensional theory described by the final action (2), we begin with a p -brane solution with a given field configuration

$$g_{\mu\nu}, \quad \phi, \quad A. \quad (3)$$

We add an extra dimension y to construct a $(p+1)$ -brane solution for the initial $(D+1)$ -dimensional action (1) as

$$\hat{A}_{\mu\dots\nu y} = A_{\mu\dots\nu}, \quad \hat{\phi}/\hat{a} = \phi/a \quad (4)$$

$$\hat{g}_{\mu\nu} = \begin{pmatrix} \exp \left[-\frac{2\tilde{d}}{(D-2)(D-1)} \frac{\phi}{a} \right] g_{\mu\nu} & 0 \\ 0 & \exp \left[\frac{2\tilde{d}}{(D-1)} \frac{\phi}{a} \right] \end{pmatrix} \quad (5)$$

with $\hat{a}^2 = [(D-1)/(D-2)]a^2 - \tilde{d}^2/(D-2)$.

Applying Poincaré duality to the field strengths in the above construction, one arrives at a distinct reduction/oxidation scheme. First one replaces the \hat{F} in the action (1) by the dual $(\tilde{d}+1)$ -form field strength $\hat{H} = e^{-\hat{a}\hat{\gamma}\hat{\phi}}\hat{\tilde{F}}$, which then satisfies $d\hat{H} = 0 = d(e^{\hat{a}\hat{\gamma}\hat{\phi}}\hat{H})$. Hence the new field strength can be defined in terms a \tilde{d} -form potential \hat{B} , *i.e.*, $\hat{H} = d\hat{B}$, and the new equations of motion will arise from the following action

$$I = \frac{1}{16\pi\hat{G}} \int d^{D+1}x \sqrt{-\hat{g}} \left[\hat{R} - \frac{\hat{\gamma}}{2}(\nabla\hat{\phi})^2 - \frac{1}{2(\tilde{d}+1)!} e^{\hat{a}\hat{\gamma}\hat{\phi}} \hat{H}^2 \right].$$

In the dimensional reduction with the dual field, any components of \hat{B} carrying a y index will vanish, *i.e.*, $\hat{B}_{\mu\cdots\nu y} = 0$. Further in the reduced theory, we have a \tilde{d} -form potential B given by

$$B_{\mu\cdots\nu} = \hat{B}_{\mu\cdots\nu},$$

and a corresponding $(\tilde{d}+1)$ -form field strength $H = dB$. The remainder of the construction is unchanged. The essential point though is that one arrives at a second independent reduction/oxidation procedure in which the form potential is completely unchanged.

Expressed in terms of the action (2), the second oxidation prescription is as follows: We begin in the D -dimensional theory described by the final action (2) with a p -brane solution given by the field configuration

$$g_{\mu\nu}, \quad \phi, \quad A. \quad (6)$$

We add an extra dimension y to construct a $(p+1)$ -brane solution for a $(D+1)$ -dimensional theory with the action

$$I = \frac{1}{16\pi\hat{G}} \int d^{D+1}x \sqrt{-\hat{g}} \left[\hat{R} - \frac{\hat{\gamma}}{2}(\nabla\hat{\phi})^2 - \frac{1}{2(d+1)!} e^{-\hat{a}\hat{\gamma}\hat{\phi}} F^2 \right] \quad (7)$$

where $\hat{a}^2 = [(D-1)/(D-2)]a^2 - d^2/(D-2)$. Also $F = dA$ is the same $(d+1)$ -form field strength that appears in the original action. The field configuration of the $(p+1)$ -brane solution leaves the nonvanishing components of the d -form potential A unchanged, and has

$$\hat{\phi}/\hat{a} = \phi/a \quad (8)$$

$$\hat{g}_{\mu\nu} = \begin{pmatrix} \exp\left[\frac{2d}{(D-2)(D-1)}\frac{\phi}{a}\right] g_{\mu\nu} & 0 \\ 0 & \exp\left[-\frac{2d}{(D-1)}\frac{\phi}{a}\right] \end{pmatrix}. \quad (9)$$

Thus beginning with p -brane solutions of one theory (2), we have two independent oxidation procedures. The first prescription in eqs. (3-5) extends the form degree of the potential A , as well as the dimension of the brane and the spacetime, while the second in eqs. (6-9) leaves the degree of the potential unchanged. Of course, the two oxidation procedures lead to different $(D+1)$ -dimensional actions.

If one repeats either of the oxidation procedures twice adding two extra dimensions, y and z , the final $(p+2)$ -brane will be isotropic in the two extra dimensions, *i.e.*, $\tilde{g}_{yy} = \tilde{g}_{zz}$

where $\tilde{g}_{\mu\nu}$ is the metric in $D + 2$ dimensions. Of course, this does not guarantee that the complete $(p + 2)$ -brane is isotropic in all of its spatial directions. The latter would depend on the details of the initial p -brane. It is the case that if one begins with an isotropic $(d - 1)$ -brane in D dimensions with an electric F charge and applies the first oxidation procedure (3-5), the resulting d -brane in $D + 1$ dimensions and the $(d + 1)$ -brane in $D + 2$ dimensions are also isotropic [3]. Hence electrically charged isotropic p -branes for theories with arbitrary dilaton coupling are readily constructed by beginning with a zero-brane or a point-like object in D dimensions carrying a conventional electric charge from a $U(1)$ two-form field strength, and applying the oxidation procedure p times to produce a p -brane in $D + p$ dimensions — see for example Appendix A and also Refs. [5, 11]. Similarly using the second oxidation procedure (6-9) and beginning with black holes carrying a $(D - 2)$ -form magnetic charge in D dimensions, one can generate the conventional magnetically charged isotropic p -branes.

If one applies the first oxidation scheme followed by the second in adding two extra dimensions, y and z , the final $(p + 2)$ -brane is not isotropic in the two extra dimensions, *i.e.*, $\tilde{g}_{yy} \neq \tilde{g}_{zz}$. Therefore multiple applications of both of these two prescriptions result in not only isotropic but also anisotropic membranes. As an example, Appendix A provides the most general extremal anisotropic brane solution for an action of the form (2).

A further comment is that in our discussion neither oxidation procedure makes any particular reference to the details of the form of the original solution in eq. (3) or (6). In particular then, there is no requirement that the initial solution be static. Thus one could extend the preceding discussion so that if one begins with a solution describing point-like objects in motion, they would be oxidized to p -brane solutions now moving in directions orthogonal to the surfaces of the branes. This observation will be essential in the discussion of p -brane scattering in the following section.

2.1 Mass and Charge Densities

It is of interest to examine how the asymptotic physical properties of the p -branes are affected in the oxidation procedures. Since the oxidized solutions are independent of the extra coordinate, their mass and charge densities will be simply related to those of the original solutions. In both prescriptions, the oxidation of the form potential is relatively trivial, hence it is straightforward to show that the corresponding charge densities are in fact unchanged. For example, applying the first oxidation procedure (3-5) to a p -brane solution with electric charge density $Q = \oint \tilde{F}$, results in a $(p + 1)$ -brane whose electric charge density is given by the same integral, *i.e.*, $\hat{Q} = \oint \hat{\tilde{F}} = \oint \tilde{F} = Q$. In other words, the charge density of the oxidized solution is given by precisely the same parameters (or combination of parameters) as that of the original solution.

A more interesting analysis is that of the mass density. Suppose that we begin with a stationary p -brane solution for the action (2) in D dimensions. We assume that we have

“Cartesian” coordinates on the background geometry

$$x^\mu = (t, x^i, y^a)$$

where t , x^i and y^a are time, \bar{p} transverse coordinates and p parallel coordinates for the brane, respectively — hence $D = p + \bar{p} + 1$. We assume that the metric is independent of t and y^a , which leads to a constant mass density. The ADM mass per unit p -volume is defined as follows [12]: Asymptotically for $r^2 = \sum_{i=1}^{\bar{p}} (x^i)^2 \rightarrow \infty$, the metric is essentially flat and so we define the deviation from a flat space as $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Now using the Cartesian coordinate metric, the mass per unit p -volume M_p is given by

$$M_p = \frac{1}{16\pi G} \oint \sum_{i=1}^{\bar{p}} n^i \left[\sum_{j=1}^{\bar{p}} (\partial_j h_{ij} - \partial_i h_{jj}) - \sum_{a=1}^p \partial_i h_{aa} \right] r^{\bar{p}-1} d\Omega$$

where n^i is a radial unit vector. If the solution is oxidized following the first prescription, we find from eq. (9)

$$\hat{h}_{\mu\nu} = \begin{pmatrix} h_{\mu\nu} - \frac{2\tilde{d}}{(D-2)(D-1)} \frac{\phi}{a} \eta_{\mu\nu} & 0 \\ 0 & \frac{2\tilde{d}}{(D-1)} \frac{\phi}{a} \end{pmatrix}$$

where we assume that $\phi \rightarrow 0$ asymptotically. Hence the mass density of the resulting $(p+1)$ -brane is

$$\begin{aligned} M_{p+1} &= \frac{1}{16\pi \hat{G}} \oint \sum_{i=1}^{\bar{p}} n^i \left[\sum_{j=1}^{\bar{p}} (\partial_j \hat{h}_{ij} - \partial_i \hat{h}_{jj}) - \sum_{\hat{a}=1}^{p+1} \partial_i \hat{h}_{\hat{a}\hat{a}} \right] r^{\bar{p}+1} d\Omega \\ &= \frac{1}{16\pi \hat{G}} \oint \sum_{i=1}^{\bar{p}} n^i \left[\sum_{j=1}^{\bar{p}} \left(\partial_j \hat{h}_{ij} - \partial_i \hat{h}_{jj} - \frac{2\tilde{d}}{(D-2)(D-1)} (\delta_{ij} \partial_j \phi/a - \delta_{jj} \partial_i \phi/a) \right) \right. \\ &\quad \left. - \sum_{a=1}^p \left(\partial_i \hat{h}_{aa} - \frac{2\tilde{d}}{(D-2)(D-1)} \delta_{aa} \partial_i \phi/a \right) - \frac{2\tilde{d}}{D-1} \partial_i \phi/a \right] r^{\bar{p}+1} d\Omega \\ &= \frac{1}{16\pi G L} \oint \sum_{i=1}^{\bar{p}} n^i \left[\sum_{j=1}^{\bar{p}} (\partial_j h_{ij} - \partial_i h_{jj}) - \sum_{a=1}^p \partial_i h_{aa} \right] r^{\bar{p}+1} d\Omega \\ &= M_p/L \quad . \end{aligned}$$

All of the explicit ϕ contributions cancel out in M_{p+1} . It is straightforward to show that the same cancellation arises when applying the second oxidation procedure. Thus in both cases, the parameters describing the mass density of the p -brane and its oxidized counterpart are identical irrespective of the details of the solutions.

One might also define Q_D , a dilaton charge density for a p -brane. A simple definition for this scalar charge density would be in terms of the asymptotic behavior of the dilaton

$$\phi \simeq -\frac{Q_D}{r^{\bar{p}-2}}$$

again assuming that $\phi \rightarrow 0$. Since both oxidation procedures yield $\hat{\phi}/\hat{a} = \phi/a$, the above definition yields $\hat{Q}_D = (\hat{a}/a) Q_D$.

3 Scattering

The existence of static multi-soliton solutions, including multi-extreme black hole and p -brane solutions, relies on the cancellation of the exchange forces generated by the scalar, form-potential and gravitational fields (the so-called “zero-force” condition). If the solitons are given velocities, however, the zero-force condition ceases to hold and dynamical, velocity-dependent forces arise. The full time-dependent equations of motion that result are highly nonlinear and in general very difficult to solve. In the absence of exact time-dependent multi-soliton solutions, Manton’s procedure [13] for the computation of a metric on the soliton moduli space yields a good low-velocity approximation for their exact dynamics. Manton’s method may be summarized as follows: One begins with a static multi-soliton solution, and gives time-dependence to the moduli characterizing the configuration. One then calculates $O(v)$ corrections to the fields by solving the constraint equations of the system with time-dependent moduli. The resulting time-dependent field configuration only satisfies the full field equations to lowest order in the velocities, and so neglects the effects of any radiation fields. Such a configuration would provide the initial data of fields and time derivatives for an exact solution. Another way of saying this is that the initial motion is tangent to the set of exact static solutions. An effective action valid to $O(v^2)$ describing the soliton motion is constructed by substituting the solution to the constraints into the field theory action. The resulting kinetic action defines a metric on the moduli space of static solutions, and the geodesic motion on this metric approximates the dynamics of the solitons. This approach was first applied to study the scattering of BPS monopoles [13], and a complete calculation of the corresponding moduli space metric and a description of its geodesics was worked out by Atiyah and Hitchin [14]. Manton’s method was subsequently adapted to general relativity by Ferrell and Eardley [15] for the study of low-velocity scattering of extreme Reissner-Nordstrom black holes.

More recently, Shiraishi [9] adapted the method of Ref. [15] to obtain the metric on moduli space for generalized multi-black hole solutions of the following action

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left[R - \frac{\gamma}{2} (\partial\phi)^2 - \frac{1}{4} e^{-a\gamma\phi} F^2 \right] \quad (10)$$

where $\gamma = 2/(D-2)$ and $F = dA$ is an ordinary two-form field strength for the vector potential A . For this theory, multi-centered solutions describing extremal electrically charged black holes have been found for arbitrary values of the dilaton-vector coupling a [16]. In these solutions, the metric may be written in isotropic coordinates as

$$ds^2 = -U^{-2}(\vec{x}) dt^2 + U^{2/(D-3)}(\vec{x}) d\vec{x}^2 \quad (11)$$

with $U(\vec{x}) = H(\vec{x})^{(D-3)/(D-3+a^2)}$ and H is a harmonic function

$$H(\vec{x}) = 1 + \sum_{i=1}^n \frac{\mu_i}{(D-3)|\vec{x} - \vec{x}_i|^{D-3}} \quad (12)$$

The vector one-form potential is

$$A = \pm \sqrt{2\beta} \frac{dt}{H(\vec{x})} \quad (13)$$

and the dilaton is given by

$$e^{-\phi/a} = H(\vec{x})^\beta \quad (14)$$

where $\beta = (D - 2)/(D - 3 + a^2)$.

For the calculation of the Manton metric in the low velocity limit, the positions \vec{x}_i are made time dependent. Then to obtain a solution which satisfies the equations of motion to $O(v)$, one must solve for the off-diagonal components of the metric (*e.g.*, g_{tx^i}) and the spatial components of the gauge field. This solution is substituted into the field theory action (with care taken to regulate various terms), and an effective lagrangian valid up to $O(v^2)$ is obtained. Shiraishi performed these calculations to produce the following effective lagrangian describing the interactions of N extremally charged black holes:

$$\begin{aligned} L = & - \sum_{i=1}^N m_i + \sum_{i=1}^N \frac{1}{2} m_i v_i^2 \\ & + \left(2 - \frac{1}{\beta}\right) \frac{2\pi G}{(A_{D-2})^2} \int d^{D-1}x H(\vec{x})^{2(\beta-1)} \sum_{i \neq j}^N m_i m_j \frac{|\vec{v}_i - \vec{v}_j|^2 \vec{r}_i \cdot \vec{r}_j}{r_i^{D-1} r_j^{D-1}} \end{aligned} \quad (15)$$

where $\vec{r}_i = \vec{x} - \vec{x}_i$, and \vec{v}_i and \vec{x}_i are, respectively, the velocity and position of the i 'th black hole. Also m_i is the mass of the i 'th black hole given by

$$m_i = \frac{A_{D-2}}{8\pi G} \beta \mu_i$$

where $A_{D-2} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)$ is the area of a unit $(D-2)$ -sphere. The first two terms in (15) correspond to the expected free particle Lagrangian to $O(v^2)$, while the remaining term is the interaction Lagrangian. As expected the interaction terms vanish as the relative velocities go to zero. In general, this contribution is highly nonlinear involving up to N -body interactions. Collecting all of the $O(v^2)$ terms yields a metric on the moduli space of these N black hole configurations [13].

The above interactions simplify for two specific values of the scalar-Maxwell coupling. For extreme $a^2 = D - 1$ black holes (*i.e.*, for which $\beta = 1/2$), the effective Lagrangian reduces to the free terms only [16, 9]

$$L = - \sum_{i=1}^N m_i + \sum_{i=1}^N \frac{1}{2} m_i v_i^2 .$$

In other words, the leading order velocity-dependent (*i.e.*, $O(v^2)$) dynamical force between the black holes is zero, and the low-velocity scattering is trivial (at least to this order). Thus one infers that the metric on the moduli space of these extreme black holes is flat.

Also for $a^2 = 1$ (*i.e.*, for $\beta = 1$), the Lagrangian (15) simplifies in that it only involves two-body interactions. Thus the effective Lagrangian is easily determined to be simply [9]

$$L = - \sum_{i=1}^N m_i + \sum_{i=1}^N \frac{1}{2} m_i v_i^2 + \frac{G}{A_{D-4}} \sum_{i \neq j}^N m_i m_j \frac{|\vec{v}_i - \vec{v}_j|^2}{|\vec{x}_i - \vec{x}_j|^{D-3}}$$

for which the scattering calculations are greatly simplified [9].

Now suppose we begin with one of Shiraishi's multi-centered solutions describing extremal electrically charged black holes, and apply the first and second oxidation procedures of section 2, p and q times, respectively — in Appendix A we perform this calculation explicitly for the static solutions in eqs. (11-14). The result will be a solution describing parallel $(p+q)$ -branes carrying an electrical charge of the $(p+1)$ -form potential for the oxidized theory in $D+p+q$ dimensions with the action

$$\bar{I} = \frac{1}{16\pi\bar{G}} \int d^{D+p+q}x \sqrt{-\bar{g}} \left[\bar{R} - \frac{\bar{\gamma}}{2} (\partial\bar{\phi})^2 - \frac{1}{2(p+2)!} e^{-\bar{a}\bar{\gamma}\bar{\phi}} \bar{F}^2 \right] \quad (16)$$

where $\bar{\gamma} = 2/(D+p+q-2)$ and $\bar{a}^2 = [(D+p+q-2)/(D-2)]a^2 - ((D-2)pq + (D-3)^2p + q)/(D-2)$. It should be clear that the dynamical interactions of these $(p+q)$ -branes are identical to that of the original black holes, and is still described by the effective lagrangian (15). This observation follows from the fact that the field-dependence of a given brane configuration is identical to that of the corresponding dimensionally reduced solution, and depends entirely on the coordinates transverse to the p -brane. The calculation which would verify this claim would begin by oxidizing the non-static black hole solutions which were explicitly calculated to leading order in the velocity expansion by Shiraishi [9]. Following the Manton method, these solutions are then to be substituted into the field theory action, but we have seen in sect. 2 that the action is unchanged by the oxidation procedure up to an overall factor of the compactification volume. Hence the effective Lagrangian density per unit $(p+q)$ -volume would be precisely the same as the Lagrangian (15) for the black holes:

$$\begin{aligned} \bar{L} = & - \sum_{i=1}^N m_i + \sum_{i=1}^N \frac{1}{2} m_i v_i^2 \\ & + \left(2 - \frac{1}{\beta} \right) \frac{2\pi\bar{G}}{(A_{D-2})^2} \int d^{D-1}x H(\vec{x})^{2(\beta-1)} \sum_{i \neq j}^N m_i m_j \frac{|\vec{v}_i - \vec{v}_j|^2 \vec{r}_i \cdot \vec{r}_j}{r_i^{D-1} r_j^{D-1}}. \end{aligned} \quad (17)$$

Now m_i is precisely the mass density of the i 'th brane with $m_i = A_{D-2}\beta\mu_i/(8\pi\bar{G})$, as per the discussion in sect. 2.1. In terms of \bar{a} , one has $\beta = (D+p+q-2)/[(p+1)(D+q-3) + \bar{a}^2]$. As a result, the parallel $(p+q)$ -branes interact in an identical manner as their dimensionally reduced point-like “descendants”. Of course, by taking the dual form potential in Shiraishi's solutions, one arrives at the same conclusion for “magnetically” charged $(p+q)$ -branes.

Just as the black hole dynamics simplified in certain special cases, so too will that of the $(p + q)$ -branes. Only two-body interactions arise for $\beta = 1$ which now corresponds to $\bar{a}^2 = 1 - p(D + q - 4)$. Here the right-hand side is always negative — recall that $D \geq 4$ and $q \geq 0$ — unless $p = 0$ in which case $\bar{a}^2 = 1$. This situation then corresponds to “isotropic” but not boost invariant q -branes carrying a conventional $U(1)$ electric charge. The other case of interest is $\beta = 1/2$ for which the $O(v^2)$ interaction vanishes. This corresponds to $\bar{a}^2 = 4 - (p - 1)(D + q - 5)$. This latter condition can be satisfied in a wide number of cases.

As one particular example consider the cases with $q = 0$ for which $\bar{a}^2 = 4 - (p - 1)(D - 5)$. These solutions correspond to a particular class of isotropic “electric” p -branes with κ -symmetry [3].⁴ These (and in fact all of the oxidized) electrically charged solutions are singular. Hence they may be regarded as arising when the fields of (2) are coupled to a source which in this particular case is governed by a $(p + 1)$ -dimensional supersymmetric sigma-model action [17] whose kinetic and Wess-Zumino terms appear with a relative coefficient fixed by the requirement of κ -symmetry. The role of the source action to the dynamics is to track the moving sources with appropriate δ -functions. Then, in a standard notation [3], such solutions have $a^2 = 2(D' - 2) - d'\tilde{d}'$, where $D' = D + p$ is the total spacetime dimension, $d' = p + 1$ is the dimension of the p -brane world-volume, and $\tilde{d}' = D' - d' - 2$ is the dimension of the world-volume of the dual brane — recall we have set $q = 0$. Following the first oxidation/reduction procedure, these κ -symmetric solutions reduce to κ -symmetric solutions, and, ultimately, to black holes in $D = D' - p$ dimensions with $a^2 = D - 1$, which, as we have seen above, were shown by Shiraishi to scatter trivially. Using his results then, we have shown that all of these κ -symmetric solutions scatter trivially in the low-velocity limit. This has been checked directly in a straightforward but tedious computation. By duality considerations, trivial scattering holds for the solitonic, “magnetic” p -branes dual to the above, singular p -branes [3].

4 Discussion

Above we have derived a general effective Lagrangian density (17) describing the slow motion scattering of arbitrary extremal p -branes. Using the the double dimensional reduction/oxidation procedures of sect. 2, we have shown that the p -brane dynamics is identical to that of the corresponding dimensionally reduced black holes. Because of our effective action is derived through reduction/oxidation, it only describes the motion of parallel p -branes moving in directions orthogonal to the surfaces of the branes. For the conventional isotropic p -branes, this in fact describes the entire moduli space of the interacting branes because of the boost invariance of the solutions in the directions parallel to the branes. For the more general anisotropic $(p + q)$ -branes considered in the Appendix, one might also consider the motion of these branes in the z^ℓ directions — *i.e.*, boosting one of these solu-

⁴Note that these solutions do not exhaust the class of all κ -symmetric p -branes, since the latter may in general involve more than one scalar and antisymmetric tensor.

tions in a direction parallel to the z^ℓ produces a new brane which will not sit in equilibrium with one of the original unboosted branes.

The most remarkable result is that to leading order in the velocity expansion the interactions vanish for all κ -symmetric p -branes within the class of solutions that we are considering. Such a flat metric on moduli space has previously been found for H -monopoles [6], fundamental strings [7] and $D = 10$ fivebranes [8], all of which fall into this class of κ -symmetric solutions. More recently, Bachas [18] has shown that for toroidal compactifications, Dirichlet-branes (see [19] and references therein) also have a flat metric.

The flat metric and consequent trivial dynamics for the κ -symmetric solutions is a somewhat surprising result, and is probably connected with the existence of flat directions in the superpotentials associated with the underlying κ -symmetric theories. Another possible answer is that only these solutions preserve half the spacetime supersymmetries in any supersymmetric embedding, and inevitably this will also constrain the dynamics considerably. For example, if we embed the four-dimensional black holes in $N = 8$, $D = 4$ supergravity, only the κ -symmetric $a = \sqrt{3}$ black hole preserves four of the spacetime supersymmetries.

The oxidation procedures of sect. 2 can be applied to produce a wide class of p -brane solutions given some family of black hole solutions to an action of the form (2) or (16) with arbitrary values of the dilaton coupling constant a . Shiraishi's solutions [16] involve *extremal* black holes with electric charge, as required for the static zero-force condition to hold. We use these as the basis for constructing a corresponding family of isotropic (and anisotropic) brane solutions in Appendix A. Gibbons [20] has found general, *i.e.*, non-extremal, static solutions describing a single black hole for the action (2), again for arbitrary a . One can just as easily apply the oxidation procedures of sect. 2 to these solutions to generate an even more general class of isotropic (and anisotropic) $(p + q)$ -brane solutions of the action (16). Typically the nonextremal "isotropic" p -branes (setting $q = 0$) will be isotropic in the spatial brane directions, but they will not have the boost invariance found in the extremal case. On the other hand, generically these nonextremal branes produced by oxidization will have a real event horizon rather than the singularities appearing in the extremal case. While in many instances these oxidized branes can be thought of as arising in a supersymmetric theory, they would usually not preserve any of the supersymmetries. For this reason, these general branes may ultimately be of less interest than the extremal solutions. Note that in the extremal case, supersymmetry alone is not enough to eliminate interactions once the branes are in motion. It is only for the maximally supersymmetric, *i.e.*, the κ -symmetric, solutions that the interactions vanish.

Note also that when the p -branes arise in supersymmetric theories, the total amount of supersymmetry preserved by a given p -brane is unaffected by the oxidation/reduction procedures we have presented. This can be seen from the fact that these procedures essentially do not change the field content of the original theory. For example, the $a = \sqrt{3}$ black hole [21] preserves half the spacetime supersymmetries in the $N = 4$, $D = 4$ theory, and continues to preserve half the supersymmetries when oxidized to an anisotropic sixbrane in the $N = 1$, $D = 10$ theory.

Acknowledgments

This research was supported by NSERC of Canada and Fonds FCAR du Québec. R.R.K. was supported by a World Laboratory Fellowship.

A Extremal (p, q) -branes

Let us begin with the action (10) and Shiraishi's black hole solutions described in eqs. (11–14). Oxidizing this system p times via the first procedure described in section 2, which then increases the form degree of the potential in each step, the action becomes

$$\hat{I} = \frac{1}{16\pi\hat{G}} \int d^{D+p}x \sqrt{-\hat{g}} \left[\hat{R} - \frac{\hat{\gamma}}{2}(\partial\hat{\phi})^2 - \frac{1}{2(p+2)!} e^{-\hat{a}\hat{\gamma}\hat{\phi}} \hat{F}^2 \right]$$

where $\hat{\gamma} = 2/(D+p-2)$ and $\hat{a}^2 = [(D+p-2)/(D-2)]a^2 - (D-3)^2p/(D-2)$. Shiraishi's black hole solutions have become p -brane solutions with the addition of p coordinates y^a running parallel to the surface of the brane. The vector potential becomes a $(p+1)$ -form potential which may be written as

$$\hat{A} = \pm \frac{\sqrt{2\beta}}{H(\vec{x})} dt dy^1 \dots dy^p$$

with the same harmonic function as in eq. (12)

$$H(\vec{x}) = 1 + \sum_{i=1}^n \frac{\mu_i}{(D-3)|\vec{x} - \vec{x}_i|^{D-3}} \quad (18)$$

and in terms of \hat{a} , $\beta = (D+p-2)/[(p+1)(D-3) + \hat{a}^2]$. Similarly the dilaton is

$$e^{-\hat{\phi}/\hat{a}} = H(\vec{x})^\beta$$

and the metric may be written

$$d\hat{s}^2 = \hat{U}^{-2}(\vec{x}) (-dt^2 + d\vec{y}^2) + \hat{U}^{2\frac{p+1}{D-3}}(\vec{x}) d\vec{x}^2$$

with $\hat{U}(\vec{x}) = H(\vec{x})^{(D-3)/[(p+1)(D-3) + \hat{a}^2]}$. These solutions are in fact within the familiar class of static “electrically” charged p -brane solutions, which display Lorentz invariance in the (t, y^a) directions [3].

Now continue oxidizing this system a further q times using the second procedure, so that this time the the degree of the potential is held fixed. The final action becomes

$$\bar{I} = \frac{1}{16\pi\bar{G}} \int d^{D+p+q}x \sqrt{-\bar{g}} \left[\bar{R} - \frac{\bar{\gamma}}{2}(\partial\bar{\phi})^2 - \frac{1}{2(p+2)!} e^{-\bar{a}\bar{\gamma}\bar{\phi}} \bar{F}^2 \right] \quad (19)$$

where $\bar{\gamma} = 2/(D + p + q - 2)$ and

$$\begin{aligned}\bar{a}^2 &= \frac{D + p + q - 2}{D + p - 2} \hat{a}^2 - \frac{(p + 1)^2 q}{D + p - 2} \\ &= \frac{D + p + q - 2}{D - 2} a^2 - \frac{(D - 2)pq + (D - 3)^2 p + q}{D - 2}.\end{aligned}$$

The final solutions are now $(p + q)$ -brane solutions with a further q coordinates z^ℓ also running parallel to the surface of the brane. The vector potential remains a $(p + 1)$ -form potential which may be written as

$$\bar{A} = \pm \frac{\sqrt{2\beta}}{H(\vec{x})} dt dy^1 \dots dy^p \quad (20)$$

again with the same harmonic function (18) while in terms of \bar{a} , one has $\beta = (D + p + q - 2)/[(p + 1)(D + q - 3) + \bar{a}^2]$. The dilaton becomes

$$e^{-\bar{\phi}/\bar{a}} = H(\vec{x})^\beta \quad (21)$$

and the metric may be written as

$$d\bar{s}^2 = \bar{U}^{-2}(\vec{x}) (-dt^2 + d\vec{y}^2) + \bar{U}^{2\frac{p+1}{D+q-3}}(\vec{x}) (d\vec{x}^2 + d\vec{z}^2) \quad (22)$$

with $\bar{U}(\vec{x}) = H(\vec{x})^{(D+q-3)/[(p+1)(D+q-3)+\bar{a}^2]}$. In this case the $(p + q)$ -brane solutions are Lorentz invariant in the (t, y^a) directions. However, this symmetry does not extend to include the coordinates z^ℓ . Note that there is no rotational symmetry between the x^i and z^ℓ directions because all of the fields and the geometry depends on x^i but not z^ℓ .

Note that there is an alternative derivation for these particular solutions as follows: If one attempts to construct a static p -brane solution of the action (19), with the fields (20-22) as an ansatz, one would find that the function H must satisfy the Laplace's equation in a $(D + q)$ -dimensional flat space described by $\{x^i, z^\ell\}$ (and also H must asymptotically approach one). If one chooses this function to be independent of the q coordinates z^ℓ , one is lead to the solution described above, and, in fact, since the entire solution is completely independent of z^ℓ , one has actually constructed a solution describing a $(p + q)$ -brane. In fact, this approach also allows for the construction of even more exotic solutions with non-parallel branes.

By taking the dual form potential above, one could also consider “magnetically” charged $(p + q)$ -branes. Completely new nonextremal anisotropic $(p + q)$ -brane solutions could be produced by applying the same construction described above to the general black holes solutions of the action (10) presented in [20]. An example of this sort of anisotropic $(p + q)$ -brane is the $D = 10$ uplifting of the (electrically or magnetically charged) H -monopole (or alternatively the uplifting of the extremally charged Kaluza-Klein black hole) [21], which has the structure of an anisotropic sixbrane with $p = 5$ and $q = 1$.

As a final note, we remark that one would refer to the above solutions as anisotropic since the y^a and z^ℓ directions are distinct. However the solutions with $p = 0$ and $q \neq 0$ are in fact isotropic in all of the spatial q -brane solutions, even though they are not boost invariant.

References

- [1] C.M. Hull and P.K. Townsend, Nucl. Phys. **B438** (1995) 109; E. Witten, Nucl. Phys. **B443** (1995) 85.
- [2] J.H. Schwarz, “The Power of M Theory” preprint hep-th/9510086; P. Hořava and E. Witten, “Heterotic and Type I String Dynamics from Eleven Dimensions” preprint hep-th/9510209.
- [3] M.J. Duff, R.R. Khuri and J.X. Lu, Phys. Rep. **259** (1995) 213.
- [4] G.T. Horowitz and A. Strominger, Nucl. Phys. **B360** (1991) 197.
- [5] M.J. Duff and J.X. Lu, Nucl. Phys. **B416** (1994) 301.
- [6] R.R. Khuri, *Phys. Lett.* **B294** (1992) 331.
- [7] R.R. Khuri, *Phys. Lett.* **B307** (1993) 302.
- [8] A.G. Felce and T.M. Samols, *Phys. Lett.* **B308** (1993) 302.
- [9] K. Shiraishi, *Nucl. Phys.* **B402** (1993) 399; “Many-body systems in Einstein-Maxwell-Dilaton theory,” preprint gr-qc/9507029.
- [10] M.J. Duff, P.S. Howe, T. Inami and K. Stelle, Phys. Lett. **B191** (1987) 70.
- [11] H. Lu, C.N. Pope, E. Sezgin, and K.S. Stelle, “Stainless Super p -branes”, preprint hep-th/9508042.
- [12] J.X. Lu, Phys. Lett. **B313** (1993) 29.
- [13] N.S. Manton, *Phys. Lett.* **B110** (1982) 54; *Phys. Lett.* **B154** (1985) 397; *Phys. Lett.* **B157** (1985) 475.
- [14] M.F. Atiyah and N.J. Hitchin, *Phys. Lett.* **A107** (1985) 21; *The Geometry and Dynamics of Magnetic Monopoles*, (Princeton University Press, 1988).
- [15] R.C. Ferrell and D.M. Eardley, *Phys. Rev. Lett.* **59** (1987) 1617; in *Frontiers in numerical relativity*, Eds. C.R. Evans, L.S. Finn and D.W. Hobill (Cambridge University Press, 1988).
- [16] K. Shiraishi, *J. Math. Phys.* **34** (1993) 1480.
- [17] J. Hughes, J. Liu and J. Polchinski, Phys. Lett. **B180** (1986) 370; E. Bergshoeff, E. Sezgin and P.K. Townsend, Phys. Lett. **B189** (1987) 75.
- [18] C. Bachas, “D-brane Dynamics”, preprint hep-th/9511043.

- [19] J. Polchinski, “Dirichlet-Branes and Ramond-Ramond Charges”, preprint hep-th/9510017.
- [20] G.W. Gibbons, Nucl. Phys. **B207** (1982) 337.
- [21] R.R. Khuri, Phys. Lett. **B259** (1991) 261; Nucl. Phys. **B387** (1992) 315; M.J. Duff, R.R. Khuri, R. Minasian and J. Rahmfeld, Nucl. Phys. **B418** (1994) 195.